

CONGRUENCES CONCERNING JACOBI POLYNOMIALS AND APÉRY-LIKE FORMULAE

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ABSTRACT. Let $p > 5$ be a prime. We prove congruences modulo p^{3-d} for sums of the general form $\sum_{k=0}^{(p-3)/2} \binom{2k}{k} t^k / (2k+1)^{d+1}$ and $\sum_{k=1}^{(p-1)/2} \binom{2k}{k} t^k / k^d$ with $d = 0, 1$. We also consider the special case $t = (-1)^d / 16$ of the former sum, where the congruences hold modulo p^{5-d} .

1. INTRODUCTION

In proving the irrationality of $\zeta(3)$, Apéry [1] mentioned the formulae

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \binom{2k}{k}^{-1} = \frac{1}{3} \zeta(2), \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1} = -\frac{2}{5} \zeta(3).$$

Later Koecher [6], and Leshchiner [7], found several analogous results for other $\zeta(r)$. In particular, in [7], by the use of some combinatorial identities it was shown that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k (2k+1)} = \frac{4}{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{3}, \quad (1)$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(-16)^k (2k+1)^2} = \frac{4}{5} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{10}. \quad (2)$$

The above evaluations are related to the power series of $\arcsin(z)$,

$$\arcsin(z) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k} z^{2k+1}}{4^k (2k+1)}, \quad |z| \leq 1, \quad (3)$$

and its integral $\int_0^z \frac{\arcsin(y)}{y} dy$, which can be expressed in closed form (see [2, Theorem 5.1]) as

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\binom{2k}{k} z^{2k+1}}{4^k (2k+1)^2} &= -\frac{i}{2} \text{Li}_2((\sqrt{1-z^2} + iz)^2) - \frac{i}{2} \arcsin^2(z) \\ &\quad + \arcsin(z) \log(2z^2 - 2iz\sqrt{1-z^2}) + \frac{i}{2} \zeta(2), \quad |z| \leq 1, \end{aligned} \quad (4)$$

where $\text{Li}_r(z) = \sum_{k=1}^{\infty} z^k / k^r$ is the polylogarithmic function.

Recently, Z. W. Sun [18], [17, p. 27] pointed out that series (3), (4) for some special values of variable admit nice finite p -analogues. He obtained the congruences for partial sums

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} t^k}{(2k+1)^{d+1}} \pmod{p^{2-d}}$$

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for $t = 1/4, 1/8, 1/16, 3/16$ if $d = 0$, and $t = 1/4$ if $d = 1$.

In this paper, we refine Sun's congruences to a higher power of p and obtain polynomial congruences for general sums of the form

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} t^k}{(2k+1)^{d+1}} \quad \text{and} \quad \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k} t^k}{k^d} \pmod{p^{3-d}}, \quad (5)$$

with $d = 0, 1$. Our approach is based on application of functional properties of Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, which are defined in terms of the Gauss hypergeometric function

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k,$$

where $(a)_0 = 1$, $(a)_k = a(a+1) \cdots (a+k-1)$, $k \geq 1$, is the Pochhammer symbol, as

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} F(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2), \quad \alpha, \beta > -1. \quad (6)$$

They satisfy the three term recurrence relation [20, Section 4.5]

$$\begin{aligned} & 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha, \beta)}(x) \\ &= [(2n+\alpha+\beta+1)(\alpha^2-\beta^2) + (2n+\alpha+\beta)_3 x]P_n^{(\alpha, \beta)}(x) \\ & \quad - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha, \beta)}(x) \end{aligned} \quad (7)$$

with the initial conditions $P_0^{(\alpha, \beta)}(x) = 1$, $P_1^{(\alpha, \beta)}(x) = (x(\alpha+\beta+2) + \alpha - \beta)/2$.

Some special cases of the Jacobi polynomials have already been used for proving many remarkable congruences containing central binomial sums. Thus, for example, the short proof of the Rodriguez-Villegas famous congruence [10]

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}, \quad (8)$$

first confirmed by Mortenson [9] via the p -adic Γ -function and the Gross-Koblitz formula, was given by Z. H. Sun [13] with the help of Legendre polynomials $P_n(x)$, which are a special case of the Jacobi polynomials: $P_n(x) = P_n^{(0,0)}(x)$. The classical Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$, corresponding to the case $\alpha = \beta = -1/2$ and $\alpha = \beta = 1/2$ in (6), respectively,

$$T_n(x) = \frac{n!}{(1/2)_n} P_n^{(-1/2, -1/2)}(x), \quad U_n(x) = \frac{(n+1)!}{(3/2)_n} P_n^{(1/2, 1/2)}(x), \quad (9)$$

were applied in many works on congruences (see, for example, [8, 16, 18, 19]). In [8], S. Mattarei and R. Tauraso used the classical Chebyshev polynomials

$$u_0(x) = 0, \quad u_n(x) = U_{n-1}(x/2) \quad (n \geq 1), \quad v_n(x) = 2T_n(x/2) \quad (n \geq 0) \quad (10)$$

to obtain congruences for the finite sums

$$\sum_{k=1}^{p-1} \frac{t^k}{k^d \binom{2k}{k}} \pmod{p^2}, \quad \sum_{k=1}^{p-1} \frac{t^{p-k} H_k(2) \binom{2k}{k}}{k^d} \pmod{p},$$

where $H_k(m) = \sum_{j=1}^k 1/j^m$ and $d = 0, 1, 2$, which are related to series expansions of even powers of arcsin and reveal some connections with infinite series for zeta values (more on zeta series and congruences related to the above sums may be found in [21, 5]).

Note that the sequences $u_n(x)$ and $v_n(x)$ given by (10) belong to a class of the so called Lucas sequences $u_n(x, y)$ and $v_n(x, y)$ which are defined by the recurrence relations

$$\begin{aligned} u_0(x, y) = 0, \quad u_1(x, y) = 1, \quad \text{and} \quad u_n(x, y) = xu_{n-1}(x, y) - yu_{n-2}(x, y) \quad \text{for } n > 1, \\ v_0(x, y) = 2, \quad v_1(x, y) = x, \quad \text{and} \quad v_n(x, y) = xv_{n-1}(x, y) - yv_{n-2}(x, y) \quad \text{for } n > 1. \end{aligned}$$

Namely, one has $u_n(x) = u_n(x, 1)$ and $v_n(x) = v_n(x, 1)$.

The paper is organized as follows. In Section 2, we prove some results concerning divisibility properties of multiple harmonic sums. In Section 3, we consider two “mixed” cases of Jacobi polynomials, $P_n^{(1/2, -1/2)}(x)$ and $P_n^{(-1/2, 1/2)}(x)$, and study their properties with application to congruences. In Section 4, we get some numerical congruences. In Section 5, by revisiting the combinatorial proof (due to D. Zagier) presented in [7, Section 5], we obtain the finite versions of identities (1) and (2), which allow us to improve significantly congruences for the first sum in (5) in the special case $t = (-1)^d/16$.

2. RESULTS CONCERNING MULTIPLE HARMONIC SUMS

We define the *multiple harmonic sum* as

$$H_n(a_1, a_2, \dots, a_r) = \sum_{0 < k_1 < k_2 < \dots < k_r \leq n} \frac{1}{k_1^{a_1} k_2^{a_2} \dots k_r^{a_r}},$$

where $n \geq r > 0$ and $(a_1, a_2, \dots, a_r) \in (\mathbb{N}^*)^r$. We also introduce the multiple sum

$$\overline{H}_n(a_1, a_2, \dots, a_r) = \sum_{0 \leq k_1 < k_2 < \dots < k_r < n} \frac{1}{(2k_1 + 1)^{a_1} (2k_2 + 1)^{a_2} \dots (2k_r + 1)^{a_r}},$$

with $(a_1, a_2, \dots, a_r) \in (\mathbb{N}^*)^r$. For the sake of brevity, if $a_1 = a_2 = \dots = a_r = a$, we write $H_n(\{a\}^r)$ and $\overline{H}_n(\{a\}^r)$. Note that both H_n and \overline{H}_n satisfy the shuffle product property:

$$\begin{aligned} H_n(a)H_n(b) &= H_n(a, b) + H_n(b, a) + H_n(a + b), \\ \overline{H}_n(a)\overline{H}_n(b) &= \overline{H}_n(a, b) + \overline{H}_n(b, a) + \overline{H}_n(a + b) \end{aligned}$$

for any positive integers n, a, b .

The values of many harmonic sums modulo a power of prime p are well known. Here is a list of results that we will need later.

(i) ([11, Theorem 5.1]) for any prime $p > r + 2$ we have

$$H_{p-1}(r) \equiv \begin{cases} -\frac{r(r+1)}{2(r+2)} p^2 B_{p-r-2} \pmod{p^3} & \text{if } r \text{ is odd,} \\ \frac{r}{r+1} p B_{p-r-1} \pmod{p^2} & \text{if } r \text{ is even;} \end{cases}$$

(ii) ([22, Theorem 3.1]) for positive integers r, s and for any prime $p > r + s$, we have

$$H_{p-1}(r, s) \equiv \frac{(-1)^s}{r + s} \binom{r + s}{s} B_{p-r-s} \pmod{p};$$

(iii) ([22, Theorem 3.5]) for positive integers r, s, t and for any prime $p > r + s + t$ such that $r + s + t$ is odd, we have

$$H_{p-1}(r, s, t) \equiv \frac{1}{2(r + s + t)} \left((-1)^r \binom{r + s + t}{r} - (-1)^t \binom{r + s + t}{t} \right) B_{p-r-s-t} \pmod{p};$$

(iv) ([21, Theorem 2.1]) for any prime $p > 5$,

$$H_{p-1}(1) \equiv -\frac{1}{2} p H_{p-1}(2) - \frac{1}{6} p^2 H_{p-1}(3) \pmod{p^5};$$

(v) ([5, Lemma 3]) for any prime $p > 5$,

$$H_{p-1}(1, 2) \equiv -3 \frac{H_{p-1}(1)}{p^2} + \frac{1}{2} p^2 B_{p-5} \pmod{p^3};$$

(vi) ([11, Theorem 5.2]) for any prime $p > r + 4$ we have

$$H_{\frac{p-1}{2}}(r) \equiv \begin{cases} -2q_p(2) + pq_p^2(2) - p^2 \left(\frac{2}{3} q_p^3(2) + \frac{7}{12} B_{p-3} \right) \pmod{p^3} & \text{if } r = 1, \\ \frac{r(2^{r+1}-1)}{2(r+1)} p B_{p-r-1} \pmod{p^2} & \text{if } r \text{ is even,} \\ -\frac{2^r-2}{r} B_{p-r} \pmod{p} & \text{if } r > 1 \text{ is odd,} \end{cases}$$

where $q_p(a) = (a^{p-1} - 1)/p$ is the so-called *Fermat quotient*.

The next three results are useful congruences involving $H_{\frac{p-1}{2}}(r)$ and $H_{\frac{p-1}{2}}(r, s)$.

Lemma 1. *Let r, a be positive integers. Then for any prime $p > r + 2$,*

$$H_{p-1}(r) \equiv H_{\frac{p-1}{2}}(r) + (-1)^r \sum_{k=0}^a \binom{r-1+k}{k} H_{\frac{p-1}{2}}(r+k) p^k \pmod{p^{a+1}}. \quad (11)$$

Moreover, if r, s are positive integers such that $r + s$ is odd, then for any prime $p > r + s$,

$$H_{\frac{p-1}{2}}(r, s) \equiv \frac{B_{p-r-s}}{2(r+s)} \left((-1)^s \binom{r+s}{s} + 2^{r+s} - 2 \right) \pmod{p}. \quad (12)$$

Proof. The congruence (11) follows immediately from the identity

$$H_{p-1}(r) = H_{\frac{p-1}{2}}(r) + (-1)^r \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^r (1 - p/k)^r}.$$

Now we show (12):

$$\begin{aligned} H_{p-1}(r, s) &= \sum_{1 \leq i < j \leq \frac{p-1}{2}} \frac{1}{i^r j^s} + \sum_{1 \leq i, j \leq \frac{p-1}{2}} \frac{1}{i^r (p-j)^s} + \sum_{1 \leq j < i \leq \frac{p-1}{2}} \frac{1}{(p-i)^r (p-j)^s} \\ &\equiv H_{\frac{p-1}{2}}(r, s) + (-1)^s H_{\frac{p-1}{2}}(r) H_{\frac{p-1}{2}}(s) + (-1)^{r+s} H_{\frac{p-1}{2}}(s, r) \\ &\equiv H_{\frac{p-1}{2}}(r, s) - H_{\frac{p-1}{2}}(s, r) \pmod{p}. \end{aligned}$$

By the shuffle product property

$$H_{\frac{p-1}{2}}(r, s) + H_{\frac{p-1}{2}}(s, r) + H_{\frac{p-1}{2}}(r+s) = H_{\frac{p-1}{2}}(r) H_{\frac{p-1}{2}}(s) \equiv 0 \pmod{p}.$$

Hence

$$2H_{\frac{p-1}{2}}(r, s) \equiv H_{p-1}(r, s) - H_{\frac{p-1}{2}}(r+s) \pmod{p}$$

and by applying (ii) and (vi) we obtain (12). \square

Theorem 1. *For any prime $p > 2$,*

$$H_{\frac{p-1}{2}}(2) + \frac{7}{6} p H_{\frac{p-1}{2}}(3) + \frac{5}{8} p^2 H_{\frac{p-1}{2}}(4) \equiv 0 \pmod{p^4}. \quad (13)$$

Proof. Let $m = \varphi(p^4) = p^3(p-1)$ and let $B_n(x)$ be the n th Bernoulli polynomial. For $r = 2, 3, 4$, Faulhaber's formula implies

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} k^{m-r} &= \frac{B_{m-r+1}\left(\frac{p+1}{2}\right) - B_{m-r+1}}{m-r+1} \\ &= \frac{B_{m-r+1}\left(\frac{p+1}{2}\right) - B_{m-r+1}\left(\frac{1}{2}\right)}{m-r+1} + \frac{B_{m-r+1}\left(\frac{1}{2}\right) - B_{m-r+1}}{m-r+1} \\ &= \sum_{k=r}^m \binom{m-r}{k-r} \frac{B_{m-k}\left(\frac{1}{2}\right)}{k-r+1} \cdot \left(\frac{p}{2}\right)^{k-r+1} + \frac{B_{m-r+1}\left(\frac{1}{2}\right) - B_{m-r+1}}{m-r+1}. \end{aligned}$$

By Euler's theorem, $H_{\frac{p-1}{2}}(r) \equiv \sum_{k=1}^{(p-1)/2} k^{m-r} \pmod{p^4}$. Since m is even, $B_{m-k} = 0$ when $m-k > 1$ and k is odd. Moreover, pB_{m-k} is p -integral by von Staudt-Clausen theorem and

$$B_{m-k}\left(\frac{1}{2}\right) = (2^{1-m+k} - 1)B_{m-k} \equiv (2^{1+k} - 1)B_{m-k} \pmod{p^4}.$$

Hence

$$\begin{aligned} \sum_{r=2}^4 \alpha_r p^{r-2} H_{\frac{p-1}{2}}(r) &\equiv \frac{p}{2} \left(\alpha_2 - \frac{6}{7} \alpha_3 \right) B_{m-2} \left(\frac{1}{2} \right) \\ &\quad + \frac{p^3}{8} (\alpha_2 - 3\alpha_3 + 4\alpha_4) B_{m-4} \left(\frac{1}{2} \right) \pmod{p^4}. \end{aligned}$$

The right-hand side vanishes if we let $\alpha_2 = 1$, $\alpha_3 = 7/6$, and $\alpha_4 = 5/8$. \square

Corollary 1. For any prime $p > 5$,

$$H_{p-1}(2) \equiv -2 \frac{H_{p-1}(1)}{p} + \frac{2}{5} p^3 B_{p-5} \pmod{p^4}, \quad (14)$$

$$H_{\frac{p-1}{2}}(2) \equiv -7 \frac{H_{p-1}(1)}{p} + \frac{17}{10} p^3 B_{p-5} \pmod{p^4}, \quad (15)$$

$$H_{\frac{p-1}{2}}(3) \equiv 6 \frac{H_{p-1}(1)}{p^2} - \frac{81}{10} p^2 B_{p-5} \pmod{p^3}, \quad (16)$$

$$H_{\frac{p-1}{2}}(1, 2) + p H_{\frac{p-1}{2}}(1, 3) \equiv -\frac{9}{2} \frac{H_{p-1}(1)}{p^2} - \frac{49}{20} p^2 B_{p-5} \pmod{p^3}. \quad (17)$$

Proof. The congruence (14) follows from (iv) and (i). By (13) and (vi) we have

$$H_{\frac{p-1}{2}}(2) + \frac{7}{6} p H_{\frac{p-1}{2}}(3) + \frac{31}{4} B_{p-5} p^3 \equiv 0 \pmod{p^4}.$$

From (11) and (vi), we deduce

$$H_{p-1}(2) \equiv 2H_{\frac{p-1}{2}}(2) + 2pH_{\frac{p-1}{2}}(3) + \frac{66}{5} B_{p-5} p^3 \pmod{p^4}.$$

By solving these two congruences with respect to $H_{\frac{p-1}{2}}(2)$ and $H_{\frac{p-1}{2}}(3)$, and by using (14), we get the result. Now we show (17). For $n = (p-1)/2$, we consider the identity

$$H_{p-1}(1, 2) = H_n(1, 2) + H_n(1) \sum_{j=1}^n \frac{1}{(p-j)^2} + \sum_{1 \leq j < i \leq n} \frac{1}{(p-i)(p-j)^2}.$$

Then by expanding the sums like in Lemma 1, we get

$$H_{p-1}(1, 2) \equiv H_n(1, 2) + H_n(1)(H_n(2) + 2p H_n(3) + 3p^2 H_n(4)) - H_n(2, 1) \\ - p(2H_n(3, 1) + H_n(2, 2)) - p^2(3H_n(4, 1) + 2H_n(3, 2) + H_n(2, 3)) \pmod{p^3}.$$

By applying the shuffle product property to $H_n(1)H_n(2)$ and $H_n(1)H_n(3)$, and by using (i), (ii) and (vi), we obtain

$$H_n(1, 2) + p H_n(3, 1) \equiv \frac{1}{2} H_{p-1}(1, 2) - \frac{1}{2} H_n(3) - \frac{27}{4} p^2 B_{p-5} \pmod{p^3}.$$

Thus the proof of (17) is complete as soon as we apply (v) and (16). \square

The relations stated in the next lemma allow us to determine $\overline{H}_{\frac{p-1}{2}}(r)$ and $\overline{H}_{\frac{p-1}{2}}(r, s)$ in terms of the multiple harmonic sums.

Lemma 2. *For any positive integers n, r we have*

$$\overline{H}_n(r) = H_{2n}(r) - \frac{H_n(r)}{2^r}.$$

Moreover, for any odd prime p and for any $r, s \geq 1$, the following congruence holds modulo p^3

$$\overline{H}_{\frac{p-1}{2}}(r, s) \equiv \frac{1}{(-2)^{r+s}} \left(H_{\frac{p-1}{2}}(s, r) + \frac{p}{2} \left(r H_{\frac{p-1}{2}}(s, r+1) + s H_{\frac{p-1}{2}}(s+1, r) \right) \right. \\ \left. + \frac{p^2}{4} \left(\binom{r+1}{2} H_{\frac{p-1}{2}}(s, r+2) + r s H_{\frac{p-1}{2}}(s+1, r+1) + \binom{s+1}{2} H_{\frac{p-1}{2}}(s+2, r) \right) \right).$$

Proof. It is easily seen that

$$H_{2n}(r) = \sum_{k=0}^{n-1} \frac{1}{(2k+1)^r} + \sum_{k=1}^n \frac{1}{(2k)^r} = \overline{H}_n(r) + \frac{H_n(r)}{2^r}.$$

The required congruence follows by expanding with respect to the powers of p the identity

$$\overline{H}_n(r, s) = \sum_{0 < j < i \leq n} \frac{1}{(2(n-i)+1)^r (2(n-j)+1)^s} = \frac{1}{(-2)^{r+s}} \sum_{0 < j < i \leq n} \frac{1}{j^s i^r \left(1 - \frac{p}{2i}\right)^r \left(1 - \frac{p}{2j}\right)^s}$$

where $n = (p-1)/2$. \square

Lemma 3. *For any prime $p > 5$,*

$$2(-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-3}{2}} \frac{(-1)^k}{2k+1} \equiv \overline{H}_{\frac{p-1}{2}}(1) - p \overline{H}_{\frac{p-1}{2}}(2) - p^2 \overline{H}_{\frac{p-1}{2}}(2, 1) \\ + p^3 \overline{H}_{\frac{p-1}{2}}(2, 2) + p^4 \overline{H}_{\frac{p-1}{2}}(2, 2, 1) \pmod{p^5}. \quad (18)$$

Proof. The following identities can be easily proved by the WZ method:

$$\sum_{k=0}^n \frac{(-16)^k \binom{n+k}{2k}}{(2k+1) \binom{2k}{k}} = 2(-1)^n \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} + \frac{1}{2n+1} \quad \text{and} \quad \sum_{k=0}^n \frac{(-16)^k \binom{n+k}{2k}}{(2k+1)^2 \binom{2k}{k}} = \frac{1}{(2n+1)^2}.$$

Let $n = (p-1)/2$. Notice that for $k \in \{0, 1, \dots, n\}$,

$$\frac{(-16)^k \binom{n+k}{2k}}{\binom{2k}{k}} = \prod_{j=0}^{k-1} \left(1 - \frac{p^2}{(2j+1)^2} \right) = \sum_{j=0}^k (-1)^j p^{2j} \overline{H}_k(\{2\}^j). \quad (19)$$

Thus, the two identities yield

$$\begin{aligned}\overline{H}_n(1) - p^2 \overline{H}_n(2, 1) + p^4 \overline{H}_n(2, 2, 1) &\equiv 2(-1)^n \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} + \frac{1 - (-1)^n 4^{p-1} \binom{2n}{n}^{-1}}{p} \pmod{p^6}, \\ \overline{H}_n(2) - p^2 \overline{H}_n(2, 2) + p^4 \overline{H}_n(2, 2, 2) &\equiv \frac{1 - (-1)^n 4^{p-1} \binom{2n}{n}^{-1}}{p^2} \pmod{p^6},\end{aligned}\tag{20}$$

and, by subtracting p times the second congruence from the first one, we get (18). \square

It is interesting to note that starting from formula (20) and then by using Lemma 2, (12), (14), (15), (i) and (vi), we easily get a generalization of Morley's congruence [3].

Corollary 2. *For any prime $p > 5$,*

$$\frac{(-1)^{(p-1)/2}}{4^{p-1}} \binom{p-1}{\frac{p-1}{2}} \equiv 1 - \frac{1}{4} p H_{p-1}(1) - \frac{1}{80} p^5 B_{p-5} \pmod{p^6}.$$

3. POLYNOMIAL CONGRUENCES

For a non-negative integer n we define the sequence

$$w_n(x) := (2n+1)F(-n, n+1; 3/2; (1-x)/2) = \frac{n!}{(1/2)_n} P_n^{(1/2, -1/2)}(x).\tag{21}$$

Using recurrence relation (7) it is easy to show that the sequence $w_n(x)$ satisfies the second order linear recurrence with constant coefficients:

$$w_{n+1}(x) = 2xw_n(x) - w_{n-1}(x)$$

and initial conditions $w_0(x) = 1$, $w_1(x) = 1 + 2x$. This implies that $w_n(x)$ has the generating function

$$W_x(z) = \sum_{n=0}^{\infty} w_n(x) z^n = \frac{1+z}{1-2xz+z^2},$$

which yields

$$w_n(x) = u_{n+1}(2x) + u_n(2x)\tag{22}$$

and

$$w_n(x) = \begin{cases} \frac{(\alpha+1)\alpha^n - (\alpha^{-1}+1)\alpha^{-n}}{\alpha - \alpha^{-1}} & \text{if } x \neq \pm 1, \\ 2n+1 & \text{if } x = 1, \\ (-1)^n & \text{if } x = -1, \end{cases}\tag{23}$$

where $\alpha = x + \sqrt{x^2 - 1}$. Note that for $x \in (-1, 1)$ we also have the alternative representation:

$$w_n(x) = \cos(n \arccos x) + \frac{x+1}{\sqrt{1-x^2}} \sin(n \arccos x).\tag{24}$$

Lemma 4. *Let $p > 3$ be a prime and $t = a/b$, where a, b are integers coprime to p . Then*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} t^k}{2k+1} \overline{H}_k(2) \equiv \frac{1}{64} \left(\frac{-1}{t} \right)^{\frac{p+1}{2}} \sum_{k=1}^{p-1} \frac{v_k(2-16t)}{k^3} \pmod{p},\tag{25}$$

$$\sum_{k=0}^{(p-1)/2} t^k \binom{2k}{k} \overline{H}_k(2) \equiv \frac{1}{2} \left(\frac{-1}{t} \right)^{\frac{p-1}{2}} \sum_{k=1}^{p-1} \frac{u_k(2-16t)}{k^2} \pmod{p}.\tag{26}$$

Proof. Let $n = (p-1)/2$. We reverse the order of summation over k :

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k} t^k}{2k+1} \overline{H}_k(2) = \sum_{k=1}^n \frac{\binom{2n-2k}{n-k} t^{n-k}}{p-2k} \overline{H}_{n-k}(2)$$

and note that by (19), for $k \in \{0, 1, \dots, n\}$,

$$\begin{aligned} \binom{2n-2k}{n-k} &= \frac{(p-1-2k)(p-2-2k) \cdots (p-(n+k))}{(n-k)!} \\ &\equiv (-1)^{n-k} \frac{(2k+1)(2k+2) \cdots (n+k)}{(n-k)!} \\ &= (-1)^{n-k} \binom{n+k}{2k} \equiv (-1)^n \frac{\binom{2k}{k}}{16^k} \pmod{p} \end{aligned}$$

and by (vi)

$$\overline{H}_{n-k}(2) = \sum_{j=k+1}^n \frac{1}{(2(n-j)+1)^2} = \sum_{j=k+1}^n \frac{1}{(p-2j)^2} \equiv \frac{H_n(2) - H_k(2)}{4} \equiv -\frac{H_k(2)}{4} \pmod{p}.$$

Hence we obtain

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k} t^k}{2k+1} \overline{H}_k(2) \equiv \frac{(-t)^n}{8} \sum_{k=1}^n \frac{\binom{2k}{k} H_k(2)}{k(16t)^k} \equiv \frac{(-t)^n}{8} \sum_{k=1}^{p-1} \frac{\binom{2k}{k} H_k(2)}{k(16t)^k} \pmod{p}.$$

Similarly, we have

$$\sum_{k=0}^n t^k \binom{2k}{k} \overline{H}_k(2) \equiv -\frac{(-t)^n}{4} \sum_{k=1}^{p-1} \frac{\binom{2k}{k} H_k(2)}{(16t)^k} \pmod{p}.$$

Now the desired congruences follow from the fact that [8, Section 6]:

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k} t^{p-k} H_k(2)}{k} \equiv -2 \sum_{k=1}^{p-1} \frac{v_k(2-t)}{k^3} \pmod{p}$$

and

$$\sum_{k=1}^{p-1} \binom{2k}{k} t^{p-k} H_k(2) \equiv -2t \sum_{k=1}^{p-1} \frac{u_k(2-t)}{k^2} \pmod{p}.$$

□

The next theorem gives us polynomial congruences for sums (5) in the case $d = 0$. Among previous results on the second sum, we mention work [14] where Z. W. Sun determined the value of $\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \pmod{p^2}$ for any integer $m \not\equiv 0 \pmod{p}$ in terms of the Lucas sequences $u_{p \pm 1}(4, m)$.

Theorem 2. *Let p be an odd prime and $t = a/b$, where a, b are integers coprime to p . Then the following congruences are true:*

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} t^k}{2k+1} &\equiv \frac{w_{\frac{p-1}{2}}(1-8t) - (-16t)^{\frac{p-1}{2}}}{p} + \frac{p^2}{64} \left(\frac{-1}{t}\right)^{\frac{p+1}{2}} \sum_{k=1}^{p-1} \frac{v_k(2-16t)}{k^3} \pmod{p^3}, \\ (-1)^{\frac{p-1}{2}} \sum_{k=0}^{(p-1)/2} \binom{2k}{k} t^k &\equiv w_{\frac{p-1}{2}}(8t-1) + \frac{p^2}{2t^{\frac{p-1}{2}}} \sum_{k=1}^{p-1} \frac{u_k(2-16t)}{k^2} \pmod{p^3}. \end{aligned}$$

Proof. Setting $n = (p-1)/2$ in (21) we get

$$w_n(x) = p \cdot F\left(\frac{1-p}{2}, \frac{1+p}{2}; \frac{3}{2}; \frac{1-x}{2}\right).$$

Observing that

$$\left(\frac{1-p}{2}\right)_k \left(\frac{1+p}{2}\right)_k = \frac{(2k)!^2}{16^k k!^2} \prod_{j=0}^{k-1} \left(1 - \frac{p^2}{(2j+1)^2}\right) \quad \text{and} \quad \left(\frac{3}{2}\right)_k = \frac{(2k+1)!}{4^k k!}, \quad (27)$$

we get the following identity:

$$w_n(x) = p \sum_{k=0}^n \frac{\binom{2k}{k}}{2k+1} \left(\frac{1-x}{8}\right)^k \prod_{j=0}^{k-1} \left(1 - \frac{p^2}{(2j+1)^2}\right). \quad (28)$$

Putting $t = (1-x)/8$ in (28) we have

$$w_n(1-8t) = p \sum_{k=0}^{n-1} \frac{\binom{2k}{k} t^k}{2k+1} \sum_{j=0}^k (-1)^j p^{2j} \overline{H}_k(\{2\}^j) + \binom{2n}{n} t^n \prod_{j=0}^{n-1} \left(1 - \frac{p^2}{(2j+1)^2}\right).$$

Using (19) for $k = n$ we obtain

$$\frac{w_n(1-8t) - (-16t)^n}{p} = \sum_{k=0}^{n-1} \frac{\binom{2k}{k} t^k}{2k+1} \sum_{j=0}^k (-1)^j p^{2j} \overline{H}_k(\{2\}^j). \quad (29)$$

Lastly, by considering the above equality modulo p^3 and by using (25), we get the first congruence of the theorem. To prove the second one, we apply the well-known symmetry property of the Jacobi polynomials

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x).$$

Then from the definition of $w_n(x)$ we get

$$w_{\frac{p-1}{2}}(x) = \frac{(-1)^{\frac{p-1}{2}} \left(\frac{p-1}{2}\right)!}{(1/2)^{\frac{p-1}{2}}} P_{\frac{p-1}{2}}^{(-1/2, 1/2)}(-x) = (-1)^{\frac{p-1}{2}} F\left(\frac{1-p}{2}, \frac{1+p}{2}; \frac{1}{2}; \frac{1+x}{2}\right). \quad (30)$$

Setting $x = 8t - 1$, by (27), we obtain

$$(-1)^{\frac{p-1}{2}} w_{\frac{p-1}{2}}(8t-1) = \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \prod_{j=1}^k \left(1 - \frac{p^2}{(2j-1)^2}\right) t^k. \quad (31)$$

Now the required congruence easily follows from (26). \square

Proposition 1. *For any non-negative integer n we have*

$$\int_0^t w_n(1-\tau^2/2) d\tau = 2 \sum_{k=0}^{n-1} \frac{(-1)^k v_{2k+1}(t)}{2k+1} + \frac{(-1)^n v_{2n+1}(t)}{2n+1}, \quad (32)$$

$$\int_0^t \frac{(-1)^n w_n(\tau^2/2-1) - 1}{\tau} d\tau = \sum_{k=1}^n \frac{(-1)^k v_{2k}(t)}{2k} - H_n(1). \quad (33)$$

Proof. From (22) it follows that

$$w_n(1 - \tau^2/2) = u_{n+1}(2 - \tau^2) + u_n(2 - \tau^2).$$

Recall that the generating functions of the sequences $u_n(t)$ and $v_n(t)$ have the form:

$$U_t(z) = \sum_{n=0}^{\infty} u_n(t) z^n = \frac{z}{z^2 - tz + 1} \quad \text{and} \quad V_t(z) = \sum_{n=0}^{\infty} v_n(t) z^n = \frac{2 - zt}{z^2 - tz + 1}.$$

Integrating corresponding power series and comparing coefficients of powers z^{2n} we have

$$\begin{aligned} \int_0^t u_n(2 - \tau^2) d\tau &= [z^{2n}] (-1)^n \int_0^t U_{2-\tau^2}(-z^2) d\tau = [z^{2n}] (-1)^n \int_0^t \frac{-z^2 d\tau}{z^4 + (2 - \tau^2)z^2 + 1} \\ &= [z^{2n}] (-1)^n \int_0^t \frac{-z^2 d\tau}{(z^2 + \tau z + 1)(z^2 - \tau z + 1)} \\ &= [z^{2n}] \frac{(-1)^n z}{2(z^2 + 1)} (\log(z^2 - tz + 1) - \log(z^2 + tz + 1)). \end{aligned}$$

Since

$$\frac{d}{dz}(\log(z^2 - tz + 1)) = \frac{2z - t}{z^2 - tz + 1} = \frac{2 - V_t(z)}{z} = - \sum_{n=1}^{\infty} v_n(t) z^{n-1}$$

it follows that

$$\log(z^2 - tz + 1) = - \sum_{n=1}^{\infty} \frac{v_n(t)}{n} z^n$$

and therefore,

$$\log(z^2 - tz + 1) - \log(z^2 + tz + 1) = -2z \sum_{n=0}^{\infty} \frac{v_{2n+1}(t)}{2n+1} z^{2n}.$$

Now formula (32) easily follows by virtue of the following relations:

$$\begin{aligned} \int_0^t u_n(2 - \tau^2) d\tau &= [z^{2n}] \frac{(-1)^{n-1} z^2}{(z^2 + 1)} \sum_{n=0}^{\infty} \frac{v_{2n+1}(t)}{2n+1} z^{2n} = [(-z)^{n-1}] \frac{1}{(z+1)} \sum_{n=0}^{\infty} \frac{v_{2n+1}(t)}{2n+1} z^n \\ &= [z^{n-1}] \frac{1}{(1-z)} \sum_{n=0}^{\infty} \frac{(-1)^n v_{2n+1}(t)}{2n+1} z^n = \sum_{k=0}^{n-1} \frac{(-1)^k v_{2k+1}(t)}{2k+1}. \end{aligned}$$

To prove (33), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{u_{2n+1}(\tau) - (-1)^n}{\tau} z^{2n} &= \frac{1}{\tau} \left(\frac{U_{\tau}(z) - U_{\tau}(-z)}{2z} - \frac{1}{z^2 + 1} \right) \\ &= \frac{z}{2(z^2 + 1)} \left(\frac{1}{z^2 - \tau z + 1} - \frac{1}{z^2 + \tau z + 1} \right). \end{aligned} \tag{34}$$

Integrating with respect to τ and resembing coefficients of z^{2n} on both sides of (34) we get

$$\begin{aligned} \int_0^t \frac{u_{2n+1}(\tau) - (-1)^n}{\tau} d\tau &= [z^{2n}] \frac{1}{z^2 + 1} \left(\log(1 + z^2) - \frac{\log(z^2 - tz + 1) + \log(z^2 + tz + 1)}{2} \right) \\ &= [z^{2n}] \frac{1}{z^2 + 1} \left(\log(1 + z^2) + \sum_{n=1}^{\infty} \frac{v_{2n}(t)}{2n} z^{2n} \right) = \sum_{k=1}^n \frac{(-1)^{k+n} v_{2k}(t)}{2k} - (-1)^n H_n(1). \end{aligned}$$

Comparing generating functions $W_{\tau^2/2-1}(z^2)$ and $(U_\tau(z) - U_\tau(-z))/(2z)$, we conclude that

$$w_n(\tau^2/2 - 1) = u_{2n+1}(\tau),$$

which completes the proof. \square

Theorem 3. *Let p be an odd prime and $t = a/b$, where a, b are integers coprime to p . Then we have the polynomial congruences*

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2} \left(\frac{t}{4}\right)^{2k} &\equiv \frac{(-1)^{\frac{p-1}{2}}(v_p(t) - t^p)}{tp^2} + \frac{2}{tp} \sum_{k=0}^{(p-3)/2} \frac{(-1)^k v_{2k+1}(t)}{2k+1} \pmod{p^2}, \\ \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \left(\frac{t}{4}\right)^{2k} &\equiv 4q_p(2) - 2pq_p^2(2) + \sum_{k=1}^{(p-1)/2} \frac{(-1)^k v_{2k}(t)}{k} \pmod{p^2}. \end{aligned}$$

Proof. Let $n = (p-1)/2$. From (28) it follows that

$$w_n(1 - \tau^2/2) = p \sum_{k=0}^n \frac{\binom{2k}{k}}{2k+1} \left(\frac{\tau}{4}\right)^{2k} \prod_{j=0}^{k-1} \left(1 - \frac{p^2}{(2j+1)^2}\right).$$

By integrating with respect to τ , and using (19) for $k = n$ we obtain

$$\begin{aligned} \int_0^t w_n(1 - \tau^2/2) d\tau &= p \sum_{k=0}^n \frac{\binom{2k}{k}}{(2k+1)^2} \cdot \frac{t^{2k+1}}{16^k} \prod_{j=0}^{k-1} \left(1 - \frac{p^2}{(2j+1)^2}\right) \\ &= p \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(2k+1)^2} \cdot \frac{t^{2k+1}}{16^k} \sum_{j=0}^k (-1)^j p^{2j} \overline{H}_k(\{2\}^j) + \frac{tp(-1)^n}{p}. \end{aligned}$$

On the other hand, by (32), we get

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(2k+1)^2} \cdot \left(\frac{t}{4}\right)^{2k} \sum_{j=0}^k (-1)^j p^{2j} \overline{H}_k(\{2\}^j) = \frac{(-1)^n(v_p(t) - t^p)}{tp^2} + \frac{2}{tp} \sum_{k=0}^{n-1} \frac{(-1)^k v_{2k+1}(t)}{2k+1}.$$

and the first congruence easily follows. To prove the second one, we note that (31) yields

$$\frac{(-1)^{\frac{p-1}{2}} w_{\frac{p-1}{2}}(\tau^2/2 - 1) - 1}{\tau} = \sum_{k=1}^{(p-1)/2} \binom{2k}{k} \prod_{j=1}^k \left(1 - \frac{p^2}{(2j-1)^2}\right) \frac{\tau^{2k-1}}{16^k}.$$

Integrating the above equality with respect to τ and applying (33) we get

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \left(\frac{t}{4}\right)^{2k} \prod_{j=1}^k \left(1 - \frac{p^2}{(2j-1)^2}\right) = \sum_{k=1}^{(p-1)/2} \frac{(-1)^k v_{2k}(t)}{k} - 2H_{\frac{p-1}{2}}(1),$$

which by (vi), implies the required congruence. \square

4. SOME NUMERICAL CONGRUENCES

The special values of the finite polylogarithms $\mathcal{L}_d(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^d}$ investigated in [8, Section 4]) allow us to give some evaluations of the polynomial congruences established in the previous section. As an example of what this means, in the next corollary we consider the first congruence of Theorem 2 when $16t \in \{1, -1, 2, 3, 1/2, 4\}$. Note that the result for $t = 1/16$ will be refined in the next section.

Corollary 3. *Let $p > 3$ be a prime. Then we have*

$$\begin{aligned}
\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)4^k} &\equiv (-1)^{\frac{p+1}{2}} \left(q_p(2) - \frac{p^2}{16} B_{p-3} \right) \pmod{p^3}, \\
\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} &\equiv \frac{(-1)^{\frac{p+1}{2}}}{36} p^2 B_{p-3} \pmod{p^3}, \\
\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} &\equiv (-1)^{\frac{p+1}{2}} \left(\frac{2}{p} \right) \left(\frac{1}{2} q_p(2) - \frac{p}{8} q_p^2(2) + \frac{p^2}{16} \left(q_p^3(2) - \frac{B_{p-3}}{8} \right) \right) \pmod{p^3}, \\
\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)} \left(\frac{3}{16} \right)^k &\equiv (-1)^{\frac{p+1}{2}} \left(\frac{3}{p} \right) \left(\frac{1}{2} q_p(3) - \frac{p}{8} q_p^2(3) + p^2 \left(\frac{q_p^3(3)}{16} - \frac{B_{p-3}}{27} \right) \right) \pmod{p^3}, \\
\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)(-32)^k} &\equiv \left(\frac{2}{p} \right) \left(2q_p(2) - p q_p^2(2) + \frac{p^2}{3} \left(2q_p^3(2) - \frac{7}{32} B_{p-3} \right) \right) \pmod{p^3}.
\end{aligned}$$

Moreover, if $p > 5$, then

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)(-16)^k} \equiv q_L - \frac{p^2}{15} \left(\frac{1}{2} q_L^3 + B_{p-3} \right) \pmod{p^3}.$$

where $q_L = (L_p - 1)/p$ is the Lucas quotient and $L_p = v_p(1, -1)$ is the p th Lucas number.

Proof. Let $n = (p-1)/2$. Setting $t = 1/4$ in the first formula of Theorem 2 and taking into account that by (23), $w_n(-1) = (-1)^n$ we get

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(2k+1)4^k} \equiv (-1)^{n+1} \left(q_p(2) + \frac{p^2}{16} \sum_{k=1}^{p-1} \frac{v_k(-2)}{k^3} \right) \pmod{p^3}.$$

Since $v_k(-2) = 2(-1)^k$, by (i) and (vi), we have

$$\sum_{k=1}^{p-1} \frac{v_k(-2)}{k^3} = 2 \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} = \frac{H_n(3)}{2} - 2H_{p-1}(3) \equiv -B_{p-3} \pmod{p},$$

which implies the first congruence of the corollary.

Similarly, setting $t = 1/16$ and using the fact that by (24), for prime $p > 3$,

$$w_{\frac{p-1}{2}}(1/2) = 2 \sin(\pi p/6) = (-1)^{\frac{p-1}{2}} \tag{35}$$

we get by [8, Section 4],

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv \frac{(-1)^{\frac{p+1}{2}}}{4} p^2 (\mathcal{L}_3(e^{i\pi/3}) + \mathcal{L}_3(e^{-i\pi/3})) \equiv \frac{(-1)^{\frac{p+1}{2}}}{36} p^2 B_{p-3} \pmod{p^3}.$$

Similarly, setting $t = -1/16$ and observing that $w_{\frac{p-1}{2}}(3/2) = L_p$ by [8, Section 4], we readily get the last congruence of the corollary. To prove the other three congruences, we note (see [8, Lemma 4.13]) that for an integer a coprime to p ,

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p} \right) \left(1 + \frac{1}{2} p q_p(a) - \frac{1}{8} p^2 q_p^2(a) + \frac{1}{16} p^3 q_p^3(a) \right) \pmod{p^4}. \tag{36}$$

From (23) and (24) we easily find that

$$w_{\frac{p-1}{2}}(0) = \cos(\pi(p-1)/4) + \sin(\pi(p-1)/4) = \sqrt{2} \sin(\pi p/4) = (-1)^{\frac{p-1}{2}} \left(\frac{2}{p}\right), \quad (37)$$

$$w_{\frac{p-1}{2}}(-1/2) = \cos(\pi(p-1)/3) + \frac{1}{\sqrt{3}} \sin(\pi(p-1)/3) = \frac{2}{\sqrt{3}} \sin(\pi p/3) = (-1)^{\frac{p-1}{2}} \left(\frac{3}{p}\right), \quad (38)$$

and

$$w_{\frac{p-1}{2}}(5/4) = 2^{\frac{p+1}{2}} - 2^{\frac{1-p}{2}}. \quad (39)$$

Now setting consequently $t = 1/8$, $t = 3/16$, and $t = -1/32$ in Theorem 2 by (36)–(39), [8, Section 4] and the formula

$$2^{\frac{1-p}{2}} \equiv \left(\frac{2}{p}\right) \left(1 - \frac{1}{2} p q_p(2) + \frac{3}{8} p^2 q_p^2(2) - \frac{5}{16} p^3 q_p^3(2)\right) \pmod{p^4},$$

we get the desired congruences. \square

By the same way, from Theorem 2 and [8, Section 4] we get the following corollary.

Corollary 4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p}\right) + \left(\frac{1}{p}\right) \frac{p^2}{24} B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k} \left(\frac{3}{16}\right)^k \equiv 1 + \left(\frac{-3}{p}\right) \frac{p^2}{12} B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}.$$

Note that the first congruence modulo p^2 in Corollary 4 appeared in [14, Corollary 1.1].

Recall that the Fibonacci numbers $\{F_n\}_{n \geq 0}$ and the Lucas numbers $\{L_n\}_{n \geq 0}$ are defined by $F_n = u_n(1, -1)$ and $L_n = v_n(1, -1)$ for all non-negative integers n .

In the next theorem we confirm two conjectures raised in [15, A90].

Theorem 4. *Let $p \neq 2, 5$ be a prime. Then we have*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} F_{2k+1}}{(2k+1)16^k} \equiv (-1)^{\frac{p+1}{2}} \frac{F_p - \left(\frac{p}{5}\right)}{p} \pmod{p^2},$$

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} L_{2k+1}}{(2k+1)16^k} \equiv (-1)^{\frac{p+1}{2}} \frac{L_p - 1}{p} \pmod{p^2}.$$

Proof. We first show that for an odd prime $p \neq 5$ and $\phi_{\pm} = (1 \pm \sqrt{5})/2$, we have

$$\phi_+ \cdot w_{\frac{p-1}{2}}(\phi_-/2) - \phi_- \cdot w_{\frac{p-1}{2}}(\phi_+/2) = (-1)^{\frac{p-1}{2}} \left(\frac{p}{5}\right) \sqrt{5}, \quad (40)$$

$$\phi_+ \cdot w_{\frac{p-1}{2}}(\phi_-/2) + \phi_- \cdot w_{\frac{p-1}{2}}(\phi_+/2) = (-1)^{\frac{p-1}{2}}. \quad (41)$$

Since $\arccos(\phi_+/2) = \pi/5$ and $\arccos(\phi_-/2) = 3\pi/5$, by (24), we get

$$\begin{aligned} & \phi_+ \cdot w_{\frac{p-1}{2}}(\phi_-/2) \mp \phi_- \cdot w_{\frac{p-1}{2}}(\phi_+/2) \\ &= 2 \left(\cos(\pi/5) \cos(3\pi(p-1)/10) \mp \cos(3\pi/5) \cos(\pi(p-1)/10) \right) \\ & \quad + \frac{\sqrt{5}}{2} \left(\frac{\sin(3\pi(p-1)/10)}{\sin(3\pi/5)} \pm \frac{\sin(\pi(p-1)/10)}{\sin(\pi/5)} \right). \end{aligned}$$

Considering primes p modulo 10 and using the well-known fact that

$$\left(\frac{p}{5}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{5} \\ -1 & \text{if } p \equiv \pm 2 \pmod{5} \end{cases}$$

we easily derive the result. Since

$$F_{2k+1} = \frac{1}{\sqrt{5}} \left(\phi_+^{2k+1} - \phi_-^{2k+1} \right),$$

from Theorem 2 we readily find that

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} F_{2k+1}}{(2k+1)16^k} &\equiv \frac{1}{p\sqrt{5}} \left(\phi_+ \cdot w_{\frac{p-1}{2}}(\phi_-/2) + (-1)^{\frac{p+1}{2}} \phi_+^p \right. \\ &\quad \left. - \phi_- \cdot w_{\frac{p-1}{2}}(\phi_+/2) - (-1)^{\frac{p+1}{2}} \phi_-^p \right) \pmod{p^2}. \end{aligned}$$

Now by (40), we get the first congruence of Theorem 4. Analogously, for the Lucas numbers we have

$$L_{2k+1} = \phi_+^{2k+1} + \phi_-^{2k+1}$$

and therefore, by Theorem 2 and (41), we obtain

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} L_{2k+1}}{(2k+1)16^k} &\equiv \frac{1}{p} \left(\phi_+ \cdot w_{\frac{p-1}{2}}(\phi_-/2) + (-1)^{\frac{p+1}{2}} \phi_+^p \right. \\ &\quad \left. + \phi_- \cdot w_{\frac{p-1}{2}}(\phi_+/2) + (-1)^{\frac{p+1}{2}} \phi_-^p \right) = \frac{(-1)^{\frac{p+1}{2}}}{p} (L_p - 1) \pmod{p^2}, \end{aligned}$$

which completes the proof. \square

Corollary 5. *Let p be an odd prime. Then we have*

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 4^k} &\equiv (-1)^{\frac{p+1}{2}} \left(\frac{1}{2} q_p^2(2) - \frac{1}{3} p q_p^3(2) - \frac{1}{16} p B_{p-3} \right) \pmod{p^2}, \\ \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k 4^k} &\equiv 2q_p(2) - p q_p^2(2) + (-1)^{\frac{p+1}{2}} 2p E_{p-3} \pmod{p^2}. \end{aligned}$$

Proof. Setting $t = 2$ in Theorem 3 and noting that $v_n(2) = 2$ for all $n \in \mathbb{N}$, we get

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 4^k} \equiv \frac{(-1)^{\frac{p+1}{2}} q_p(2)}{p} + \frac{2}{p} \sum_{k=0}^{(p-3)/2} \frac{(-1)^k}{2k+1} \pmod{p^2}.$$

Now the required congruence easily follows from Lemmas 3 and 2, and formulae (12) and (vi). Similarly, we have

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k 4^k} \equiv 4q_p(2) - 2p q_p^2(2) + 2 \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \pmod{p^2}.$$

Observing that by [12, Corollary 3.3] and (vi),

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} = H_{\lfloor p/4 \rfloor} - H_{(p-1)/2} \equiv -q_p(2) + \frac{1}{2} p q_p^2(2) + (-1)^{\frac{p+1}{2}} p E_{p-3} \pmod{p^2},$$

we conclude the proof. \square

5. TWO REMARKABLE SPECIAL CASES

By revisiting the combinatorial proof (due to D. Zagier) presented in [7, Section 5], it is easy to obtain the finite versions of identities (1), (2) in the following form: if r is a positive odd integer then

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{16^k} \left(\sum_{j=0}^{\frac{r-1}{2}} \frac{(-1)^j \overline{H}_k(\{2\}^j)}{(2k+1)^{r-2j}} - \frac{(-1)^{\frac{r-1}{2}}}{4} \cdot \frac{\overline{H}_k(\{2\}^{\frac{r-1}{2}})}{(2k+1)} \right) \\ = \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)^r} + \frac{(-1)^{\frac{r-1}{2}}}{4} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{16^k \binom{n+k}{2k+1}} \cdot \frac{(-1)^{n-k} \overline{H}_k(\{2\}^{\frac{r-1}{2}})}{(2k+1)}, \end{aligned} \quad (42)$$

and if r is a positive even integer then

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(-16)^k} \left(\sum_{j=0}^{\frac{r}{2}-1} \frac{(-1)^j \overline{H}_k(\{2\}^j)}{(2k+1)^{r-2j}} + \frac{(-1)^{\frac{r}{2}-1}}{4} \cdot \frac{\overline{H}_k(\{2\}^{\frac{r}{2}-1})}{(2k+1)^2} \right) \\ = \sum_{k=0}^{n-1} \frac{1}{(2k+1)^r} + \frac{(-1)^{\frac{r}{2}-1}}{4} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(-16)^k \binom{n+k}{2k+1}} \cdot \frac{\overline{H}_k(\{2\}^{\frac{r}{2}-1})}{(2k+1)^2}. \end{aligned} \quad (43)$$

Theorem 5. For any prime $p > 5$ we have

$$\sum_{k=0}^{\frac{p-3}{2}} \frac{\binom{2k}{k}}{16^k (2k+1)} \equiv (-1)^{\frac{p-1}{2}} \left(\frac{H_{p-1}(1)}{12} + \frac{3}{160} p^4 B_{p-5} \right) \pmod{p^5}, \quad (44)$$

$$\sum_{k=0}^{\frac{p-3}{2}} \frac{\binom{2k}{k}}{(-16)^k (2k+1)^2} \equiv \frac{H_{p-1}(1)}{5p} + \frac{7}{20} p^3 B_{p-5} \pmod{p^4}. \quad (45)$$

Proof. According to (19), it follows that

$$\begin{aligned} \frac{\binom{2k}{k}}{(-16)^k \binom{n+k}{2k+1}} &= \frac{(2k+1) \binom{2k}{k}}{(-16)^k (n-k) \binom{n+k}{2k}} \equiv - \frac{2}{\left(1 - \frac{p}{2k+1}\right) (1 - p^2 \overline{H}_k(2) + p^4 \overline{H}_k(2, 2))} \\ &\equiv -2 \left(1 + \frac{p}{2k+1} + \left(\frac{1}{(2k+1)^2} + \overline{H}_k(2) \right) p^2 + \left(\frac{1}{(2k+1)^3} + \frac{\overline{H}_k(2)}{2k+1} \right) p^3 \right. \\ &\quad \left. + \left(\frac{1}{(2k+1)^4} + \frac{\overline{H}_k(2)}{(2k+1)^2} + \overline{H}_k(4) + \overline{H}_k(2, 2) \right) p^4 \right) \pmod{p^5}. \end{aligned} \quad (46)$$

Therefore, by (42) for $r = 1$, (46) and (18), we get

$$\begin{aligned} \frac{3(-1)^n}{2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{16^k (2k+1)} &\equiv -2p \overline{H}_n(2) - p^2 (\overline{H}_n(3) + 2\overline{H}_n(2, 1)) - p^3 \overline{H}_n(4) \\ &\quad - p^4 (\overline{H}_n(5) + \overline{H}_n(2, 3) + \overline{H}_n(4, 1)) \pmod{p^5}. \end{aligned} \quad (47)$$

Now, by Lemma 2 we can replace the terms \overline{H}_n with the corresponding expressions involving H_{p-1} and H_n . So the right-hand side of (47) becomes

$$\begin{aligned} & -2pH_{p-1}(2) - p^2H_{p-1}(3) - p^3H_{p-1}(4) - p^4H_{p-1}(5) \\ & + \frac{p}{2}H_n(2) + \frac{p^2}{8}(H_n(3) + 2H_n(1, 2)) + \frac{p^3}{16}(H_n(4) + 4H_n(1, 3) + 2H_n(2, 2)) \\ & - \frac{p^4}{32}(H_n(5) + 4H_n(2, 3) + 3H_n(3, 2) + 7H_n(1, 4)) \pmod{p^5}. \end{aligned}$$

Since

$$H_n(2, 2) = \frac{1}{2}(H_n(2)^2 - H_n(4)) \equiv -\frac{31}{5}pB_{p-5} \pmod{p^2},$$

by (i), (ii), and (vi), the above expression simplifies to

$$-2pH_{p-1}(2) + \frac{1}{2}pH_n(2) + \frac{1}{8}p^2H_n(3) + \frac{1}{4}p^2(H_n(1, 2) + pH_n(1, 3)) + \frac{513}{320}p^4B_{p-5} \pmod{p^5}.$$

Finally, we apply (14), (15), (16) and (17) to obtain

$$\frac{H_{p-1}(1)}{8} + \frac{9}{320}p^4B_{p-5} \pmod{p^5}$$

which concludes our proof of (44).

As regards (45), by (43) for $r = 2$, and (46), we have

$$\begin{aligned} 5 \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} & \equiv 4\overline{H}_n(2) - 2(\overline{H}_n(2) + p\overline{H}_n(3) + p^2(\overline{H}_n(4) + \overline{H}_n(2, 2)) \\ & + p^3(\overline{H}_n(5) + \overline{H}_n(2, 3))) \pmod{p^4}. \end{aligned} \quad (48)$$

As before, after replacing the terms \overline{H}_n , the right-hand side of (48) becomes

$$\begin{aligned} & 2H_{p-1}(2) - 2pH_{p-1}(3) - 2p^2H_{p-1}(4) - 2p^3H_{p-1}(5) \\ & - \frac{1}{2}H_n(2) + \frac{p}{4}H_n(3) + \frac{p^2}{8}(H_n(4) - H_n(2, 2)) \\ & + \frac{p^3}{16}(H_n(5) - 2H_n(2, 3) - H_n(3, 2)) \pmod{p^4}. \end{aligned}$$

By (i), (ii), and (vi), it simplifies to

$$2H_{p-1}(2) - \frac{1}{2}H_n(2) + \frac{1}{4}pH_n(3) + \frac{9}{4}p^3B_{p-5} \pmod{p^4}.$$

By using (14), (15) and (16), we get

$$\frac{H_{p-1}(1)}{p} + \frac{7}{4}p^3B_{p-5} \pmod{p^4}$$

and (45) is established. \square

Note that the congruence (44) as well as (45) modulo p^3 in Theorem 5 were first conjectured by Z. W. Sun in [18, Conjecture 5.1]. From (44) and (29) we easily get the following congruence, which was proposed in [15, A32].

Corollary 6. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}\overline{H}_k(2)}{16^k(2k+1)} \equiv (-1)^{\frac{p-1}{2}} \frac{H_{p-1}(1)}{12p^2} \pmod{p^2}.$$

Note that the value of the corresponding infinite series is known (see [4, p. 230–231])

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k} \overline{H}_k(2)}{16^k(2k+1)} = \frac{\pi^3}{648}.$$

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